MATH 4030 Problem Set 5[1]

Due date: Nov 19, 2019

Reading assignment: do Carmo's Section 4.3, 4.4

Problems: (Those marked with † are optional.)

1. Compute all the Christoffel symbols of a surface of revolution parametrized by
$$X(u,v) = (f(v)\cos u, f(v)\sin u, g(v)), \quad (u,v) \in (0,2\pi) \times \mathbb{R}$$
where $f, g : \mathbb{R} \to \mathbb{R}$ are smooth functions with $f > 0$ everywhere.

2. Let $\alpha(t) : (-\epsilon, \epsilon) \to S$ be a curve on a surface $S \subset \mathbb{R}^3$. Suppose $X(t), Y(t)$ are two tangential vector fields defined along the curve α. Prove that
$$\frac{d}{dt}\langle X(t), Y(t)\rangle = \langle \nabla_{\alpha'(t)} X(t), Y(t)\rangle + \langle X(t), \nabla_{\alpha'(t)} Y(t)\rangle.$$
Using this result, prove that the angle between two parallel vector fields X, Y along a curve is always constant.

3. Suppose $X(u,v) : U \to S \subset \mathbb{R}^3$ is an *orthogonal* parametrization of a surface S such that the first fundamental form is diagonal:
$$(g_{ij}) = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix}.$$
Show that the Gauss curvature is given by
$$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right].$$
If, in addition, that X is an *isothermal* parametrization, i.e. $E = G = \lambda(u,v)$, then show that
$$K = -\frac{1}{2\lambda}\Delta(\log\lambda)$$
where $\Delta := \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ is the standard Euclidean Laplace operator.

4. Does there exist a parametrization $X(u,v) : U \to \mathbb{R}^3$ of a surface S such that the first and second fundamental forms are given by:
   (a) $(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $(h_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$;
   (b) $(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u \end{pmatrix}$ and $(h_{ij}) = \begin{pmatrix} \cos^2 u & 0 \\ 0 & 1 \end{pmatrix}$?

   Explain your answer.

5. (a) Let p be a point on a surface $S \subset \mathbb{R}^3$. Prove that $K(p) > 0$ if and only if there exists a point $p_0 \in \mathbb{R}^3$ such that p is a local maximum of the function $f(x) = |x - p_0|^2$.
   (b) Show that there is no compact surface $S \subset \mathbb{R}^3$ with $K \leq 0$ everywhere.

6. (†) Prove that the surfaces parametrized by $(u,v) \in (0,+\infty) \times (0,2\pi)$,
$$X(u,v) = (u\cos v, u\sin v, \log u)$$
$$\tilde{X}(u,v) = (u\cos v, u\sin v, v)$$
have the same Gauss curvature at the points $X(u,v)$ and $\tilde{X}(u,v)$. However, show that the map $\tilde{X} \circ X^{-1}$ is not an isometry.

7. (†) Explain why the saddle surface $\{z = x^2 - y^2\}$ is not locally isometric to any round sphere or cylinder.

[1] Last revised on October 30, 2019